When is a symmetric pin-jointed framework isostatic?

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ABSTRACT

Maxwell's rule from 1864 gives a necessary condition for a framework to be isostatic in 2D or in 3D. Given a framework with point group symmetry, group representation theory is exploited to provide further necessary conditions. This paper shows how, for an isostatic framework, these conditions imply very simply stated restrictions on the numbers of those structural components that are unshifted by the symmetry operations of the framework. In particular, it turns out that an isostatic framework in 2D can belong to one of only six point groups. Some conjectures and initial results are presented that would give sufficient conditions (in both 2D and 3D) for a framework that is realized generically for a given symmetry group to be an isostatic framework.

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1. Introduction

This paper deals with *isostatic* frameworks, i.e., pin-jointed bar assemblies, commonly referred to in engineering literature as truss structures, that are both kinematically and statically determinate. Such systems are minimally infinitesimally rigid and maximally stress-free: they can be termed 'just rigid'. Our ultimate goal is to answer the question posed in the title: when are symmetric pin-jointed frameworks isostatic? As a first step, the present paper provides a series of *necessary* conditions obeyed by isostatic frameworks that possess symmetry, and also summarizes conjectures and initial results on *sufficient* conditions.

Frameworks provide a model that is useful in applications ranging from civil engineering (Graver, 2001) and the study of granular materials (Donev et al., 2004) to biochemistry (Whiteley, 2005). Many of these model frameworks have symmetry. In applications, both practical and theoretical advantages accrue when the framework is isostatic. In a number of applications, point symmetry of the framework appears naturally, and it is therefore of interest to understand the impact of symmetry on the rigidity of the framework.

Maxwell (1864) formulated a necessary condition for infinitesimal rigidity, a counting rule for 3D pin-jointed structures, with an obvious counterpart in 2D; these were later refined by Calladine (1978). Laman (1970) provided sufficient criteria for infinitesimal rigidity in 2D, but there are well known problems in extending this to 3D (Graver et al., 1993).

The Maxwell counting rule, and its extensions, can be re-cast to take account of symmetry (Fowler and Guest, 2000) using the language of point-group representations (see e.g., Bishop, 1973). The symmetry-extended Maxwell rule gives additional information from which it has often been possible to detect and explain 'hidden' mechanisms and states of self-stress in cases where the standard counting rules give insufficient information (Fowler and Guest, 2002, 2005; Schulze, 2008). Similar symmetry extensions have been derived for other classical counting rules (Ceulemans and Fowler, 1991; Guest and Fowler, 2005).

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In the present paper, we will show that the symmetry-extended Maxwell rule can be used to provide necessary conditions for a finite framework possessing symmetry to be stress-free and infinitesimally rigid, i.e., isostatic. It turns out that symmetric isostatic frameworks must obey some simply stated restrictions on the counts of structural components that are fixed by various symmetries. For 2D systems, these restrictions imply that isostatic structures must have symmetries belonging to one of only six point groups. For 3D systems, all point groups are possible, as convex triangulated polyhedra (isostatic by the theorems of Cauchy and Dehn (Cauchy, 1813; Dehn, 1916)) can be constructed in all groups (Section 3.2), although restrictions on the placement of structural components may still apply.

For simplicity in this presentation, we will restrict our configurations to realisations in which all joints are distinct. Thus, if we consider an abstract representation of the framework as a graph, with vertices corresponding to joints, and edges corresponding to bars, then we are assuming that the mapping from the graph to the geometry of the framework is injective on the vertices. Complications can arise in the non-injective situation, and will be considered separately (Schulze, 2008).

The structure of the paper is as follows: Maxwell's rule, and its symmetry-extended version, are introduced in Section 2, where a symmetry-extended version of a necessary condition for a framework to be isostatic is given, namely the equisymmetry of the representations for mechanisms and states of self-stress. In Section 3 the calculations are carried out in 2D, leading to restrictions on the symmetries and configurations of 2D isostatic frameworks, and in 3D, leading to restrictions on the placement of structural components with respect to symmetry elements. In Section 4 we conjecture sufficient conditions for a framework realized generically for a symmetry group to be isostatic, both in the plane and in 3D.

2. Background

2.1. Scalar counting rule

Maxwell's rule (Maxwell, 1864) in its modern form (Calladine, 1978),

$$m - s = 3j - b - 6, \tag{1}$$

expresses a condition for the determinacy of an unsupported, three-dimensional pin-jointed frame, in terms of counts of structural components. In Eq. (1), b is the number of bars, j is the number of joints, m is the number of infinitesimal internal mechanisms and s is the number of states of self-stress. A statically determinate structure has $s = 0$; a kinematically determinate structure has $m = 0$; isostatic structures have $s = m = 0$.

The form of (1) arises from a comparison of the dimensions of the underlying vector spaces that are associated with the equilibrium, or equivalently the compatibility, relationships for the structure (Pellegrino and Calladine, 1986).

Firstly, the equilibrium relationship can be written as

$$\mathbf{At} = \mathbf{f},$$

where $\mathbf{A}$ is the *equilibrium* matrix; $\mathbf{t}$ is a vector of internal bar forces (tensions), and lies in a vector space of dimension b; $\mathbf{f}$ is an assignment of externally applied forces, one to each joint, and, as there are $3j$ possible force components, $\mathbf{f}$ lies in a vector space of dimension $3j$ (this vector space is the tensor product of a j-dimensional vector space resulting from assigning a scalar to each joint, and a three-dimensional vector space in which a 3D force vector can be defined). Hence $\mathbf{A}$ is a $3j \times b$ matrix.

A state of self-stress is a solution to $\mathbf{At} = \mathbf{0}$, i.e., a vector in the nullspace of $\mathbf{A}$; if $\mathbf{A}$ has rank r, the dimension of this nullspace is

$$s = b - r. \tag{2}$$

Further, the compatibility relationship can be written as

$$\mathbf{Cd} = \mathbf{e},$$

where $\mathbf{C}$ is the *compatibility* matrix; $\mathbf{e}$ is a vector of infinitesimal bar extensions, and lies in a vector space of dimension b; $\mathbf{d}$ is a vector of infinitesimal nodal displacements, there are $3j$ possible nodal displacements and so $\mathbf{d}$ lies in a vector space of dimension $3j$. Hence $\mathbf{C}$ is a $b \times 3j$ matrix. In fact, it is straightforward to show (see e.g., Pellegrino and Calladine, 1986) that $\mathbf{C}$ is identical to $\mathbf{A}^\mathrm{T}$. The matrix $\mathbf{C}$ is closely related to the rigidity matrix commonly used in the mathematical literature: the rigidity matrix is formed by multiplying each row of $\mathbf{C}$ by the length of the corresponding bar. Of particular relevance here is that fact that the rigidity matrix and $\mathbf{C}$ have an identical nullspace.

A *mechanism* is a solution to $\mathbf{A}^\mathrm{T}\mathbf{d} = 0$, i.e., a vector in the left-nullspace of $\mathbf{A}$, and the dimension of this space is $3j - r$. However, this space has a basis comprised of m internal mechanisms and six rigid-body mechanisms, and hence

$$m + 6 = 3j - r. \tag{3}$$

Eliminating r from (2) and (3) recovers Maxwell's Eq. (1).

The above derivation assumes that the system is three-dimensional, but it can be applied to two-dimensional frameworks, simply replacing $3j - 6$ by $2j - 3$:

$$m - s = 2j - b - 3. \tag{4}$$

2.2. Symmetry-extended counting rule

The scalar formula (1) has been shown (Fowler and Guest, 2000) to be part of a more general symmetry version of Maxwell's rule. For a framework with point group symmetry $\mathscr{G}$,

$$3D: \quad \Gamma(m) - \Gamma(s) = \Gamma(j) \times \Gamma_{xyz} - \Gamma(b) - \Gamma_{xyz} - \Gamma_{R_x R_y R_z}, \tag{5}$$

where each Γ is known in applied group theory as a *representation* of $\mathscr{G}$ (Bishop, 1973), or in mathematical group theory as a *character* (James and Liebeck, 2001). For any set of objects q, $\Gamma(q)$ can be considered as a vector, or ordered set, of the traces of the transformation matrices $\mathbf{D}_q(R)$ that describe the transformation of q under each symmetry operation R that lies in $\mathscr{G}$. In this way, (5) may be considered as a set of equations, one for each class of symmetry operations in $\mathscr{G}$. Alternatively, and equivalently, each $\Gamma(q)$ can be written as the sum of irreducible representations/characters of $\mathscr{G}$ (Bishop, 1973). In (5) the various sets q are sets of bars b, joints j, mechanisms m and states of self-stress s; Γ_{xyz} and $\Gamma_{R_x R_y R_z}$ are the translational and rotational representations, respectively. Calculations using (5) can be completed by standard manipulations of the character table of the group (Atkins et al., 1970; Bishop, 1973; Altmann and Herzig, 1994).

The restriction of (5) to two-dimensional systems (assumed to lie in the xy-plane) is made by replacing Γ_{xyz} with Γ_{xy} and $\Gamma_{R_x R_y R_z}$ with Γ_{R_z}, as appropriate to the reduced set of rigid-body motions.

$$2D: \quad \Gamma(m) - \Gamma(s) = \Gamma(j) \times \Gamma_{xy} - \Gamma(b) - \Gamma_{xy} - \Gamma_{R_z}. \tag{6}$$

Examples of the application of (5) and (6), with detailed working, can be found in Fowler and Guest (2000, 2002, 2005), and further background, giving explicit transformation matrices, will be found in Kangwai and Guest (2000).

In the context of the present paper, we are interested in isostatic systems, which have $m = s = 0$, and hence obey the symmetry condition $\Gamma(m) = \Gamma(s) = 0$. In fact, the symmetry Maxwell equation (5) and (6) gives the necessary condition $\Gamma(m) - \Gamma(s) = 0$, as it cannot detect the presence of paired equisymmetric mechanisms and states of self-stress.

The symmetry-extended Maxwell equation corresponds to a set of k scalar equations, where k is the number of irreducible representations of $\mathscr{G}$ (the number of rows in the character table), or equivalently the number of conjugacy classes of $\mathscr{G}$ (the number of columns in the character table). The former view has been used in previous papers; the latter will be found useful in the present paper for deriving restrictions on isostatic frameworks.

2.3. The need for restrictions

That existence of symmetry typically imposes restrictions on isostatic frameworks can be seen from some simple general considerations. Consider a framework having point-group symmetry $\mathscr{G}$. Suppose that we place all bars and joints freely (so that no bar or joint is mapped onto itself by any symmetry operation). Both b and j must then be multiples of $|\mathscr{G}|$, the order of the group: $b = \bar{b}|\mathscr{G}|, j = \bar{j}|\mathscr{G}|$. Can such a framework be isostatic? Any isostatic framework obeys the scalar Maxwell rule with $m - s = 0$ as a necessary condition. In three dimensions, we have $b = 3j - 6$, and hence:

$$3D: \quad \bar{b} = 3\bar{j} - \frac{6}{|\mathscr{G}|}.$$

In two dimensions, we have $b = 2j - 3$, and hence:

$$2D: \quad \bar{b} = 2\bar{j} - \frac{3}{|\mathscr{G}|}.$$

As $\bar{b}$ and $\bar{j}$ are integers, $|\mathscr{G}|$ is restricted to values 1, 2, 3 and 6 in 3D, and 1 and 3 in 2D. Immediately we have that if the point group order is not one of these special values, it is impossible to construct an isostatic framework with all structural components placed freely: any isostatic framework with $|\mathscr{G}| \neq 1, 3$ (2D) or $|\mathscr{G}| \neq 1, 2, 3, 6$ (3D) must have some components in *special positions* (components that are unshifted by some symmetry operation).

In the Schoenflies notation (Bishop, 1973), the point groups of orders 1, 2, 3 and 6 are

$$
\begin{aligned}
|\mathscr{G}| = 1: \quad & \mathscr{C}_1, \\
|\mathscr{G}| = 2: \quad & \mathscr{C}_2, \mathscr{C}_s, \mathscr{C}_i, \\
|\mathscr{G}| = 3: \quad & \mathscr{C}_3, \\
|\mathscr{G}| = 6: \quad & \mathscr{C}_{3h}, \mathscr{C}_3, \mathscr{D}_3, \mathscr{S}_6.
\end{aligned}
$$

Further restrictions follow from the symmetry-adapted Maxwell rules (5) and (6). In a hypothetical framework where all bars and joints are placed freely, the bar and joint representations are

$$\Gamma(b) = \bar{b}\Gamma_{\text{reg}}; \quad \Gamma(j) = \bar{j}\Gamma_{\text{reg}}.$$

where Γ_{reg} is the *regular representation* of $\mathscr{G}$ with trace $|\mathscr{G}|$ under the identity operation, and 0 under all other operations. The representations Γ_{xyz} and Γ_{xy} have trace 3 and 2, respectively, under the identity operation, and hence Eqs. (5) and (6) become

$$3\text{D}: \quad \Gamma(m) - \Gamma(s) = 3\bar{j}\Gamma_{\mathrm{reg}} - \bar{b}\Gamma_{\mathrm{reg}} - \Gamma_{xyz} - \Gamma_{R_x R_y R_z},$$

$$2\text{D}: \quad \Gamma(m) - \Gamma(s) = 2\bar{j}\Gamma_{\mathrm{reg}} - \bar{b}\Gamma_{\mathrm{reg}} - \Gamma_{xy} - \Gamma_{R_z},$$

which can be written as,

$$3\text{D}: \quad \Gamma(m) - \Gamma(s) = \left(3\bar{j} - \bar{b} - \frac{6}{|\mathscr{G}|}\right)\Gamma_{\mathrm{reg}} + \left(\frac{6}{|\mathscr{G}|}\Gamma_{\mathrm{reg}} - \Gamma_{xyz} - \Gamma_{R_x R_y R_z}\right),$$

$$2\text{D}: \quad \Gamma(m) - \Gamma(s) = \left(2\bar{j} - \bar{b} - \frac{3}{|\mathscr{G}|}\right)\Gamma_{\mathrm{reg}} + \left(\frac{3}{|\mathscr{G}|}\Gamma_{\mathrm{reg}} - \Gamma_{xy} - \Gamma_{R_z}\right).$$

If we have arranged that our hypothetical framework has satisfied the scalar Maxwell rule, we are left with a 'discrepancy term', given by

$$3\text{D}: \quad \Gamma(m) - \Gamma(s) = \left(\frac{6}{|\mathscr{G}|}\Gamma_{\mathrm{reg}} - \Gamma_{xyz} - \Gamma_{R_x R_y R_z}\right),$$

$$2\text{D}: \quad \Gamma(m) - \Gamma(s) = \left(\frac{3}{|\mathscr{G}|}\Gamma_{\mathrm{reg}} - \Gamma_{xy} - \Gamma_{R_z}\right).$$

Thus in both 2D and 3D, our hypothetical framework cannot be isostatic unless the rigid-body motions span a multiple of the regular representation, when the discrepancy term will disappear. Within groups of the specified orders, this term disappears only for: in 3D, $\{\mathscr{C}_1, \mathscr{C}_s, \mathscr{C}_i, \mathscr{C}_3, \mathscr{C}_{3h}, \mathscr{C}_3, \mathscr{S}_6\}$, and in 2D, $\{\mathscr{C}_1, \mathscr{C}_3\}$. Thus, for example, 3D frameworks of $\mathscr{C}_2$ or $\mathscr{D}_3$ symmetry, with all structural components shifted by all symmetry operations, cannot be isostatic, even when they satisfy the scalar Maxwell count: in both cases, evaluation of the discrepancy term shows that the hypothetical 3D framework would have a totally symmetric mechanism and a state of self-stress that is antisymmetric with respect to 2-fold rotation.

Frameworks of higher symmetry, such as the icosahedral ($|\mathscr{G}| = 120$ or 60) or cubic groups ($|\mathscr{G}| = 48, 24$ or 12) cannot satisfy even the isostatic scalar Maxwell count without having structural components in special positions.

Thus, even this simple example shows that for many groups some restriction on positions of bars and points is imposed by symmetry, and implies that symmetry adds extra necessary conditions for frameworks to be isostatic.

3. Derivation of conditions for isostatic frameworks

In order to apply (5) to any particular framework, we require, in addition to the standard Γ_{xyz} and $\Gamma_{R_x R_y R_z}$, a knowledge of the bar and joint permutation representations: $\Gamma(b)$ and $\Gamma(j)$. In other words, for each symmetry operation in $\mathscr{G}$, we need to determine the numbers of bars and joints that remain unshifted by that operation. It is necessary to perform this count only once per conjugacy class.

Setting $\Gamma(m) - \Gamma(s)$ to zero in (5) and (6), class by class, will give up to k independent necessary conditions for the framework to be isostatic. We will carry out this procedure once and for all point groups, as there is a very limited set of possible operations to consider. The two-dimensional and three-dimensional cases will be considered separately.

3.1. Two-dimensional isostatic frameworks

In this section we treat the two-dimensional case: bars, joints, and their associated displacements are all confined to the plane (Note that frameworks that are isostatic in the plane may have out-of-plane mechanisms when considered in 3-space.). The relevant symmetry operations are: the identity (E), rotation by $2\pi/n$ about a point (C_n), and reflection in a line (σ). The possible groups are the groups $\mathscr{C}_n$ and $\mathscr{C}_n$ for all natural numbers n. $\mathscr{C}_n$ is the cyclic group generated by C_n, and $\mathscr{C}_n$ is generated by a $\{C_n, \sigma\}$ pair. The group $\mathscr{C}_1$ is usually called $\mathscr{C}_s$.

All two-dimensional cases can be treated in a single calculation, as shown in Table 1. Each entry in the table is the trace (character) of the appropriate representation (indicated in the left column) of the symmetry (indicated in the top line). Characters are calculated for four operations: we distinguish C_2 from the C_n operation with $n > 2$. Each line in the table represents a stage in the evaluation of (6). Similar tabular calculations are found in Fowler and Guest (2000) and subsequent papers.

Table 1
Calculations of characters for representations for the 2D symmetry-extended Maxwell Eq. (6).

	E	$C_{n>2}$	C_2	σ
$\Gamma(j)$	j	j_c	j_c	j_σ
$\times \Gamma_{xy}$	-2	$2\cos\phi$	-2	-0
$= \Gamma(j) \times \Gamma_{xy}$	$2j$	$2j_c\cos\phi$	$-2j_c$	-0
$-\Gamma(b)$	$-b$	0	$-b_2$	$-b_\sigma$
$-(\Gamma_{xy} + \Gamma_{R_z})$	-3	$-2\cos\phi - 1$	1	1
$= \Gamma(m) - \Gamma(s)$	$2j - b - 3$	$2(j_c - 1)\cos\phi - 1$	$-2j_c - b_2 + 1$	$-b_\sigma + 1$

To treat all two-dimensional cases in a single calculation, we need a notation that keeps track of the fate of structural components under the various operations, which in turn depends on how the joints and bars are placed with respect to the symmetry elements. The notation used in Table 1 is as follows:

j is the total number of joints;
j_c is the number of joints lying on the point of rotation ($C_{n>2}$ or C_2) (note that, as we are considering only cases where all joints are distinct, $j_c = 0$ or 1);
j_σ is the number of joints lying on a given mirror line;
b is the total number of bars;
b_2 is the number of bars left unshifted by a C_2 operation (see Fig. 1(a) and note that C_n with $n > 2$ shifts all bars);
b_σ is the number of bars unshifted by a given mirror operation (see Fig. 1(b): the unshifted bar may lie in, or perpendicular to, the mirror line).

Each of the counts refers to a particular symmetry element and any structural component may therefore contribute to one or more count, for instance, a joint counted in j_c also contributes to j_σ for each mirror line present.

From Table 1, the symmetry treatment of the 2D Maxwell equation reduces to scalar equations of four types. If $\Gamma(m) - \Gamma(s) = 0$, then

$$E : \quad 2j - b = 3, \tag{7}$$
$$C_2 : \quad 2j_c + b_2 = 1, \tag{8}$$
$$\sigma : \quad b_\sigma = 1, \tag{9}$$
$$C_{n>2} : \quad 2(j_c - 1)\cos\phi = 1, \tag{10}$$

where a given equation applies when the corresponding symmetry operation is present in $\mathscr{G}$. Some observations on 2D isostatic frameworks, arising from this set of equations are:

(i) Trivially, all 2D frameworks have the identity element and (7) simply restates the scalar Maxwell rule (4) with $m - s = 0$.
(ii) Presence of a C_2 element imposes limitations on the placement of bars and joints. As both j_c and b_2 must be non-negative integers, (8) has the unique solution $b_2 = 1, j_c = 0$. In other words, an isostatic 2D framework with a C_2 element of symmetry has no joint on the point of rotation, but exactly one bar centered at that point.
(iii) Similarly, presence of a mirror line implies, by (9), that $b_\sigma = 1$ for that line, but places no restriction on the number of joints in the same line, and hence allows this bar to lie either in, or perpendicular to, the mirror.
(iv) Deduction of the condition imposed by a rotation of higher order $C_{n>2}$ proceeds as follows. Eq. (10) with $\phi = 2\pi/n$ implies

$$(j_c - 1)\cos\left(\frac{2\pi}{n}\right) = \frac{1}{2} \tag{11}$$

and as j_c is either 0 or 1, this implies that $j_c = 0$ and $n = 3$. Thus, a 2D isostatic framework cannot have a C_n rotational element with $n > 3$, and when either a C_2 or C_3 rotational element is present, no joint may lie at the centre of rotation.

In summary, a 2D isostatic framework may have only symmetry operations drawn from the list $\{E, C_2, C_3, \sigma\}$, and hence the possible symmetry groups $\mathscr{G}$ are six in number: $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_s, \mathscr{C}_2, \mathscr{C}_3$. Group by group, the conditions necessary for a 2D framework to be isostatic are then as follows:

$\mathscr{C}_1$: $b = 2j - 3$.
$\mathscr{C}_2$: $b = 2j - 3$ with $b_2 = 1$ and $j_c = 0$, and as all other bars and joints occur in pairs, j is even and b is odd.
$\mathscr{C}_3$: $b = 2j - 3$ with $j_c = 0$, and hence all joints and bars occur in sets of 3.
$\mathscr{C}_s$: $b = 2j - 3$ with $b_\sigma = 1$ and all other bars occurring in pairs. Symmetry does not restrict j_σ.
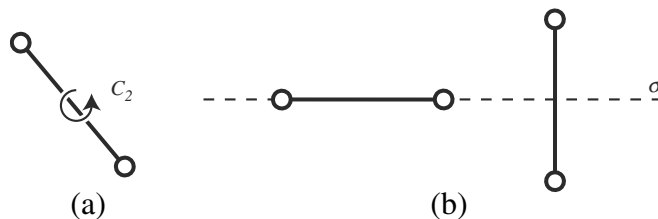

Fig. 1. Possible placement of a bar with respect to a symmetry element in two dimensions, such that it is unshifted by the associated symmetry operation: (a) C_2 centre of rotation; (b) mirror line.

$\mathscr{C}_2$: $b = 2j - 3$ with $j_c = 0$ and $b_2 = b_\sigma = 1$. A central bar lies in one of the two mirror lines, and perpendicular to the other. Any additional bars must lie in the general position, and hence occur in sets of 4, with joints in sets of 2 and 4. Hence b is odd and j is even.

$\mathscr{C}_3$: $b = 2j - 3$ with $j_c = 0$ and $b_\sigma = 1$ for each of the three mirror lines.

We consider whether these condition are also sufficient in Section 4.1.

Fig. 2 gives examples of small 2D isostatic frameworks for each of the possible groups, including cases where bars lie in, and perpendicular to, mirror lines.

3.2. Three-dimensional isostatic frameworks

The families of possible point groups of 3D objects are: the icosahedral $\mathscr{I}, \mathscr{I}_h$; the cubic $\mathscr{T}, \mathscr{T}_h, \mathscr{T}_d, \mathscr{O}, \mathscr{O}_h$; the axial $\mathscr{C}_n, \mathscr{C}_{nh}, \mathscr{C}_n$; the dihedral $\mathscr{D}_n, \mathscr{D}_{nh}, \mathscr{D}_{nd}$; the cyclic $\mathscr{S}_{2n}$; and the trivial $\mathscr{C}_s, \mathscr{C}_i, \mathscr{C}_1$ (Atkins et al., 1970). The relevant symmetry operations are: proper rotation by $2\pi/n$ about an axis, C_n, and improper rotation, S_n (C_n followed by reflection in a plane perpendicular to the axis). By convention, the identity $E \equiv C_1$, inversion $i \equiv S_2$, and reflections $\sigma \equiv S_1$ are treated separately.

The calculation is shown in Table 2. Characters are calculated for six operations. For proper rotations, we distinguish E and C_2 from the C_n operations with $n > 2$. For improper rotations, we distinguish σ and i from the $S_{n>2}$ operations. We exclude from consideration the degenerate case of a single bar, and assume that the total number of joints is greater than three.

The notation used in Table 2 is

j is the total number of joints;
j_n is the number of joints lying on the C_n axis;
j_c is the number of joints (0 or 1) lying on the unique central point (if any). Such joints are unshifted by all operations;
j_σ is the number of joints lying on a given σ mirror-plane;
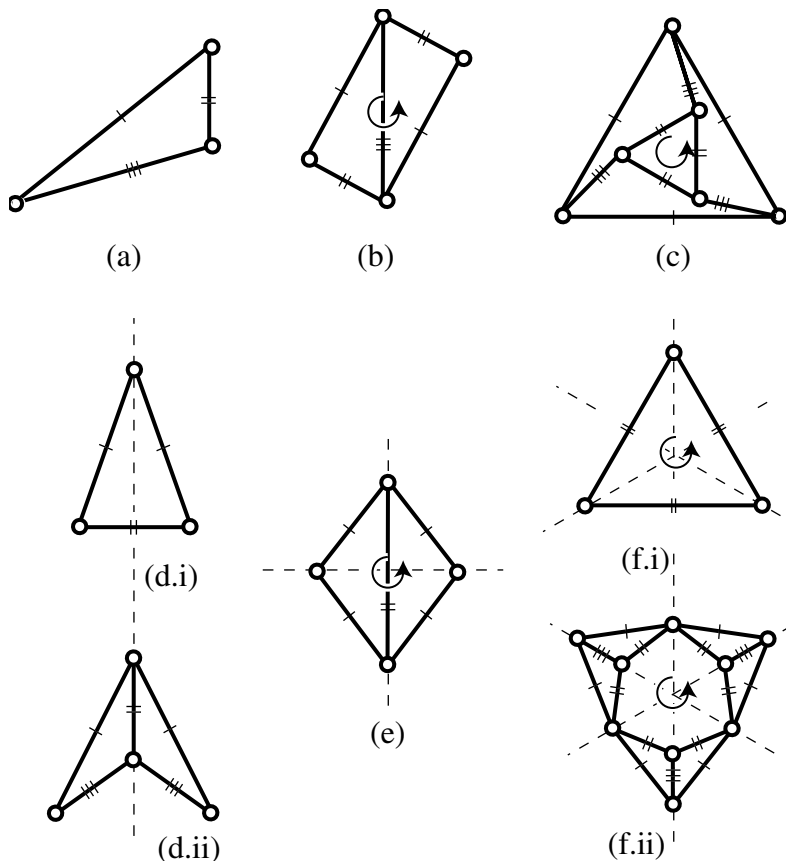

Fig. 2. Examples, for each of the possible groups, of small 2D isostatic frameworks, with bars which are equivalent under symmetry marked with the same symbol: (a) $\mathscr{C}_1$; (b) $\mathscr{C}_2$; (c) $\mathscr{C}_3$; (d) $\mathscr{C}_s \equiv \mathscr{C}_{1v}$; (e) $\mathscr{C}_{2v}$; (f) $\mathscr{C}_{3v}$. Mirror lines are shown dashed, and rotation axes are indicated by a circular arrow. For each of $\mathscr{C}_s$ and $\mathscr{C}_{3v}$, two examples are given: (i) where each mirror has a bar centered at, and perpendicular to, the mirror line; (ii) where a bar lies in each mirror line. For $\mathscr{C}_{2v}$, the bar lying at the centre must lie in one mirror line, and perpendicular to the other.

Table 2
Calculations of characters for representations in the 3D symmetry-extended Maxwell Eq. (5).

	E	$C_{n \neq 2}(\phi)$	C_2	σ	i	$S_{n \neq 2}(\phi)$
$\Gamma(j)$	j	j_n	j_2	j_σ	j_c	j_c
$\times \Gamma_{xyz}$	3	$2\cos\phi + 1$	-1	-1	-3	$2\cos\phi - 1$
$= \Gamma(j) \times \Gamma_{xyz}$	$3j$	$(2\cos\phi + 1)j_n$	$-j_2$	j_σ	$-3j_c$	$(2\cos\phi - 1)j_c$
$-\Gamma(b)$	$-b$	$-b_n$	$-b_2$	$-b_\sigma$	$-b_c$	$-b_{nc}$
$-(\Gamma_{xyz} + \Gamma_{R_x R_y R_z})$	-6	$-4\cos\phi - 2$	2	0	0	0
$= \Gamma(m) - \Gamma(s)$	$3j - b - 6$	$(2\cos\phi + 1)(j_n - 2) - b_n$	$-j_2 - b_2 + 2$	$j_\sigma - b_\sigma$	$-3j_c - b_c$	$(2\cos\phi - 1)j_c - b_{nc}$

b is the total number of bars;

b_n is the number of bars unshifted by a $C_{n>2}$ rotation: note that each such bar must lie along the axis of the rotation (see Fig. 3(a));

b_{nc} is the number of bars unshifted by the improper rotation $S_{n>2}$: note that such bars must lie along the axis of the rotation, and be centered on the central point of the group (see Fig. 4(a));

b_c is the number of bars unshifted by the inversion i: note that the centre of the bar must lie at the central point of the group, but no particular orientation is implied (see Fig. 4(b));

b_2 is the number of bars unshifted by the C_2 rotation: such bars must lie either along, or perpendicular to and centered on, the axis (see Fig. 3(a) and (b));

b_σ is the number of bars unshifted by a given σ mirror operation (see Fig. 5(a) and (b)).

Again, each of the counts refers to a particular symmetry element, and so, for instance the joint counted in j_c also contributes to j, j_n and j_σ.

From Table 2, the symmetry treatment of the 3D Maxwell equation reduces to scalar equations of six types. If $\Gamma(m) - \Gamma(s) = 0$, then

$$E: \quad 3j - b = 6, \tag{12}$$
$$\sigma: \quad b_\sigma = j_\sigma, \tag{13}$$
$$i: \quad 3j_c + b_c = 0, \tag{14}$$
$$C_2: \quad j_2 + b_2 = 2, \tag{15}$$
$$C_{n>2}: \quad (j_n - 2)(2\cos\phi + 1) = b_n, \tag{16}$$
$$S_{n>2}: \quad j_c(2\cos\phi - 1) = b_{nc}, \tag{17}$$

where a given equation applies when the corresponding symmetry operation is present in $\mathscr{G}$.

Some observations on 3D isostatic frameworks, arising from the above, are

(i) From (12), the framework must satisfy the scalar Maxwell rule (1) with $m - s = 0$.
(ii) From (13), each mirror that is present contains the same number of joints as bars that are unshifted under reflection in that mirror.
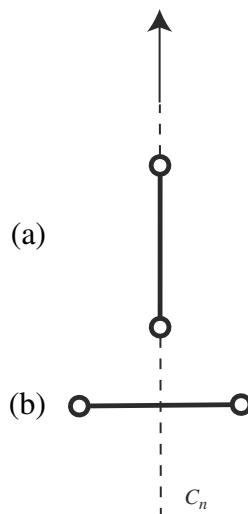
Fig. 3. Possible placement of a bar unshifted by a proper rotation about an axis: (a) for any $C_{n \geqslant 2}$; (b) for C_2 alone.
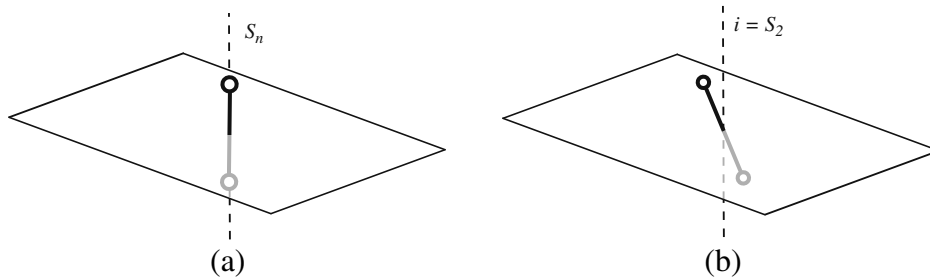
Fig. 4. Possible placement of a bar unshifted by an improper rotation about an axis: (a) for any $S_{n \geqslant 2}$; (b) for $i = S_2$.
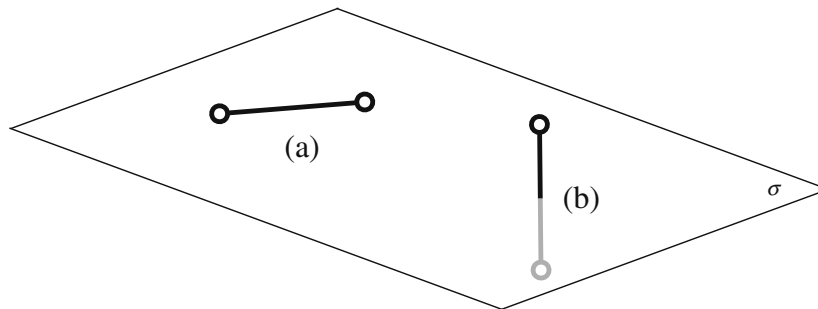

Fig. 5. Possible placement of a bar unshifted by a reflection in a plane: (a) lying in the plane; (b) lying perpendicular to the plane.

(iii) From (14), a centro-symmetric framework has neither a joint nor a bar centered at the inversion centre.

(iv) For a C_2 axis, (15) has solutions

$$(j_2, b_2) = (2, 0), (1, 1), (0, 2).$$

The count b_2 refers to both bars that lie along, and those that lie perpendicular to, the axis. However, if a bar were to lie along the C_2 axis, it would contribute 1 to b_2 and 2 to j_2 thus generating a contradiction of (15), so that in fact all bars included in b_2 must lie perpendicular to the axis.

(v) Eq. (16) can be written, with $\phi = 2\pi/n$, as

$$(j_n - 2)\left(2\cos\left(\frac{2\pi}{n}\right) + 1\right) = b_n,$$

with $n > 2$. The non-negative integer solution $j_n = 2, b_n = 0$, is possible for all n. For $n > 2$ the factor $(2\cos(2\pi/n) + 1)$ is rational at $n = 3, 4, 6$, but generates a further distinct solution only for $n = 3$:

$n = 3$

$$0(j_3 - 2) = b_3,$$

and so here $b_3 = 0$, but j_3 is unrestricted.

$n = 4$

$$j_4 - 2 = b_4,$$

C_4 implies $C_4^2 = C_2$ about the same axis, and hence $b_4 = 0$, and $j_4 = j_2 = 2$.

$n = 6$

$$2(j_6 - 2) = b_6,$$

C_6 implies $C_6^3 = C_2$ and $C_6^2 = C_3$ about the same axis, and hence $b_6 = b_3 = 0$, and $j_6 = j_3 = j_2 = 2$. Thus b_n is 0 for any $n > 2$, and only in the case $n = 3$ may j_n depart from 2.

(vi) Likewise, Eq. (17) can be written, with $\phi = 2\pi/n$, as

$$\left(2\cos\left(\frac{2\pi}{n}\right) - 1\right)j_c = b_{nc},$$

with $n > 2$. The integer solution $j_c = 0, b_{nc} = 0$, is possible for all n. For $n > 2$ the factor $(2\cos(2\pi/n) - 1)$ is rational at $n = 3, 4, 6$, but generates no further solutions:

$n = 3$

$$-2j_c = b_{3c},$$

and so $j_c = b_{3c} = 0$.
$n = 4$

$$-j_c = b_{4c},$$

and so $j_c = b_{4c} = 0$.
$n = 6$

$$0j_c = b_{6c},$$

and $b_{6c} = 0$ but S_6 implies $S_6^3 = i$ and hence also $j_c = 0$.

(vii) For a framework with icosahedral ($\mathscr{I}$ or $\mathscr{I}_h$) symmetry, the requirement that $j_5 = 2$ for each 5-fold axis implies that the framework must include a single orbit of 12 vertices that are the vertices of an icosahedron. Similarly, for a framework with a $\mathcal{O}$ or $\mathcal{O}_h$ symmetry, the requirement that $j_4 = 2$ implies that the framework must include a single orbit of 6 vertices that are the vertices of an octahedron.

In contrast to the 2D case, in 3D the symmetry conditions do not exclude any point group. For example, a fully triangulated convex polyhedron, isostatic by the Theorem of Cauchy and Dehn (Cauchy, 1813; Dehn, 1916) can be constructed to realize any 3D point group. Beginning with the regular triangulated polyhedra (the tetrahedron, octahedron, icosahedron), infinite families of isostatic frameworks can be constructed by expansions of these polyhedra using operations of truncation and capping. For example, to generate isostatic frameworks with only the rotational symmetries of a given triangulated polyhedron, we can 'cap' each face with a twisted octahedron, consistent with the rotational symmetries of the underlying polyhedron: the resultant polyhedron will be an isostatic framework with the rotational symmetries of the underlying polyhedron, but none of the reflection symmetries. An example of the capping of a regular octahedron is shown in Fig. 6. Similar techniques can be applied to create polyhedra for any of the point groups.

One interesting possibility arises from consideration of groups that contain C_3 axes. Eq. (16) allows an unlimited number of joints, though not bars, along a 3-fold symmetry axis. Thus, starting with an isostatic framework, joints may be added symmetrically along the 3-fold axes. To preserve the Maxwell count, each additional joint is accompanied by 3 new bars. Thus, for instance, we can 'cap' every face of an icosahedron to give the compound icosahedron-plus-dodecahedron (the second stellation of the icosahedron), as illustrated in Fig. 7, and this process can be continued ad infinitum adding a pile of 'hats' consisting of a new joint, linked to all three joints of an original icosahedral face (Fig. 8). Similar constructions starting from cubic and trigonally symmetric isostatic frameworks can be envisaged. Addition of a single 'hat' to a triangle of a framework is one of the Hennenberg moves (Tay and Whiteley, 1985): changes that can be made to an isostatic framework without introducing extra mechanisms or states of self-stress.

(a)   (b)
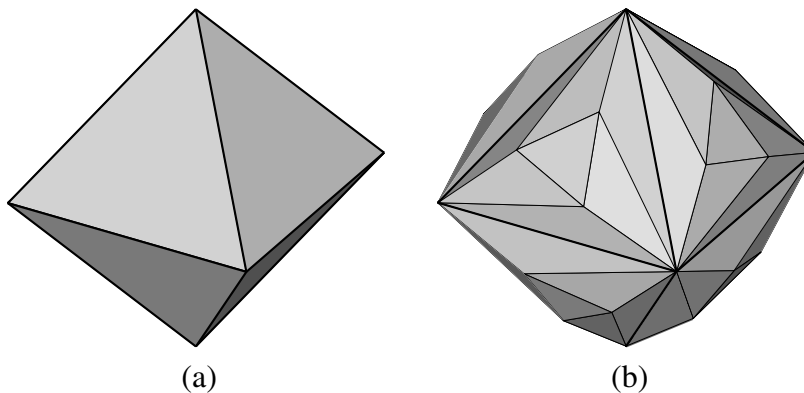
Fig. 6. A regular octahedron (a), and a convex polyhedron (b) generated by adding a twisted octahedron to every face of the original octahedron. The polyhedron in (b) has the rotation but not the reflection symmetries of the polyhedron in (a). If a framework is constructed from either polyhedron by placing bars along edges, and joints at vertices, the framework will be isostatic.
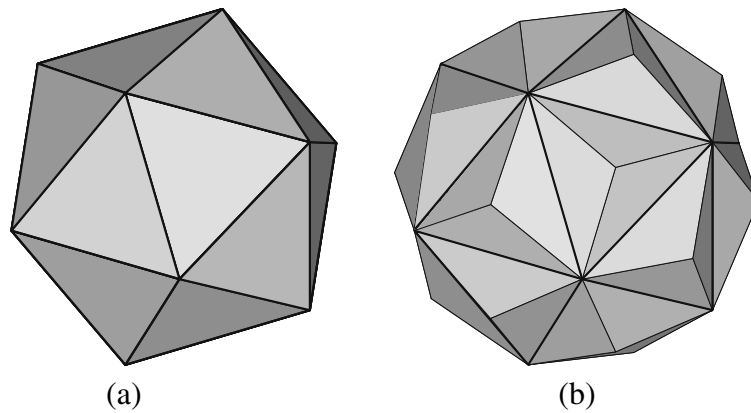
Fig. 7. An icosahedron (a), and the second stellation of the icosahedron (b). If a framework is constructed from either polyhedron by placing bars along edges, and joints at vertices, the framework will be isostatic. The framework (b) could be constructed from the framework (a) by 'capping' each face of the original icosahedron preserving the C_{3v} site symmetry.

4. Sufficient conditions for isostatic realisations

4.1. Conditions for two-dimensional isostatic frameworks

For a framework with point-group symmetry $\mathscr{G}$ the previous section has provided some necessary conditions for the realization to be isostatic. These conditions included some over-all counts on bars and joints, along with sub-counts on special classes of bars and joints (bars on mirrors or perpendicular to mirrors, bars centered on the axis of rotation, joints on the centre of rotation etc.). Here, assuming that the framework is realized with the joints in a configuration as generic as possible (subject to the symmetry conditions), we investigate whether these conditions are sufficient to guarantee that the framework is isostatic.

The simplest case is the identity group ($\mathscr{C}_1$). For this basic situation, the key result is Laman's Theorem. In the following, we take $G = \{J, B\}$ to define the connectivity of the framework, where J is the set of j joints and B the set of b bars, and we take p to define the positions of all of the joints in 2D.

Theorem 1. *(Laman, 1970) For a generic configuration in 2D, p, the framework G(p) is isostatic if and only if $G = \{J, B\}$ satisfies the conditions:*

(i) $b = 2j - 3$;
(ii) *for any non-empty set of bars B^*, which contacts just the joints in J^*, with $|B^*| = b^*$ and $|J^*| = j^*$, $b^* \leqslant 2j^* - 3$.*

Our goal is to extend these results to other symmetry groups. With the appropriate definition of 'generic' for symmetry groups (Schulze, 2008), we can anticipate that the necessary conditions identified in the previous sections for the corresponding group plus the Laman condition identified in Theorem 1, which considers subgraphs that are not necessarily symmetric, will be sufficient. For three of the plane symmetry groups, this has been confirmed. We use the previous notation for the point groups and the identification of special bars and joints, and describe a configuration as 'generic with symmetry group $\mathscr{G}$' if, apart from conditions imposed by symmetry, the points are in a generic position (the constraints imposed by the local site symmetry may remove 0,1 or 2 of the two basic freedoms of the point).

Theorem 2. *(Schulze, in preparation-a) If p is a plane configuration generic with symmetry group $\mathscr{G}$, and G(p) is a framework realized with these symmetries, then the following necessary conditions are also sufficient for G(p) to be isostatic: $b = 2j - 3$ and for any non-empty set of bars $B^*, b^* \leqslant 2j^* - 3$ and*

(i) *for $C_s : b_\sigma = 1$;*
(ii) *for $C_2 : b_2 = 1, j_c = 0$;*
(iii) *for $C_3 : j_c = 0$.*

For the remaining groups, we have a conjecture.

Conjecture 1. *If p is a plane configuration generic with symmetry group $\mathscr{G}$, and G(p) is a framework realized with these symmetries, then the following necessary conditions are also sufficient for G(p) to be isostatic: $b = 2j - 3$ and for any non-empty set of bars $B^*, b^* \leqslant 2j^* - 3$ and*
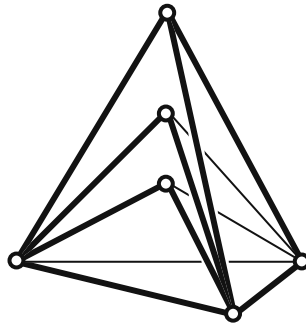
Fig. 8. A series of 'hats' added symmetrically along a 3-fold axis of an isostatic framework leaves the structure isostatic.

    (i) *for $C_2 : b_2 = 1$ and $b_\sigma = 1$ for each mirror;*
    (ii) *for $C_3 : j_c = 0$ and $b_\sigma = 1$ for each mirror.*

An immediate consequence of this theorem (and the conjecture) is that there is (would be) a polynomial time algorithm to determine whether a given framework in generic position modulo the symmetry group $\mathscr{G}$ is isostatic. Although the Laman condition of Theorem 1 involves an exponential number of subgraphs of G, there are several algorithms that determine whether it holds in *cjb* steps where *c* is a constant. The pebble game (Hendrickson and Jacobs, 1997) is an example. The additional conditions for being isostatic with the symmetry group $\mathscr{G}$ trivially can be verified in constant time.

4.2. Conditions for isostatic 3D frameworks

In 3D, there is no known counting characterization of generically isostatic frameworks, although we have the necessary conditions: $3j - b - 6 = 0$ and $3j' - b' - 6 \geqslant 0$ for all subgraphs with $j' \geqslant 3$ (Graver, 2001). There are, however a number of constructions for graphs which are known to be generically isostatic in 3D (see e.g., Tay and Whiteley, 1985; Whiteley, 1991). If we assume that we start with such a graph G, then it is natural to ask whether the additional necessary conditions for a realization $G(p)$ that is generic with point group symmetry $\mathscr{G}$ to be isostatic are also sufficient. In contrast to the plane case, where we only needed to state these conditions once, for the entire graph, in 3D for all subgraphs G' of G whose realizations $G'(p)$ are symmetric with a subgroup $\mathscr{G}'$ of $\mathscr{G}$, with the full count $3j' - b' - 6 = 0$, we need to assert the conditions corresponding to the symmetry operations in $\mathscr{G}'$ as well. These conditions are clearly necessary, and for all reflections, half-turns, and 6-fold rotations in $\mathscr{G}'$, they do not follow from the global conditions on the entire graph (as they would in the plane). See Schulze (in preparation-b) for details.

All of the above conditions combined, however, are still not sufficient for a three-dimensional framework $G(p)$ which is generic with point group symmetry $\mathscr{G}$ to be isostatic, because even if $G(p)$ satisfies all of these conditions, the symmetry imposed by $\mathscr{G}$ may force parts of $G(p)$ to be 'flattened' so that a self-stress of $G(p)$ is created. For more details on how 'flatness' caused by symmetry gives rise to additional necessary conditions for three-dimensional frameworks to be isostatic, we refer the reader to Schulze et al. (in preparation).

References

Altmann, S.L., Herzig, P., 1994. Point-group Theory Tables. Clarendon Press, Oxford.
Atkins, P.W., Child, M.S., Phillips, C.S.G., 1970. Tables for Group Theory. Oxford University Press, Oxford.
Bishop, D.M., 1973. Group Theory and Chemistry. Clarendon Press, Oxford.
Calladine, C.R., 1978. Buckminster Fuller's 'Tensegrity' structures and Clerk Maxwell's rules for the construction of stiff frames. International Journal of Solids and Structures 14, 161–172.
Cauchy, A.L., 1813. Recherche sur les polyèdres – premier mémoire. Journal de l'Ecole Polytechnique 9, 66–86.
Ceulemans, A., Fowler, P.W., 1991. Extension of Euler's theorem to the symmetry properties of polyhedra. Nature 353, 52–54.
Dehn, M., 1916. Über die Starreit konvexer Polyeder. Mathematische Annalen 77, 466–473.
Donev, A., Torquato, S., Stillinger, F.H., Connelly, R., 2004. Jamming in hard sphere and disk packings. Journal of Applied Physics 95, 989–999.
Fowler, P.W., Guest, S.D., 2000. A symmetry extension of Maxwell's rule for rigidity of frames. International Journal of Solids and Structures 37, 1793–1804.
Fowler, P.W., Guest, S.D., 2002. Symmetry and states of self stress in triangulated toroidal frames. International Journal of Solids and Structures 39, 4385–4393.
Fowler, P.W., Guest, S.D., 2005. A symmetry analysis of mechanisms in rotating rings of tetrahedra. Proceedings of the Royal Society: Mathematical, Physical & Engineering Sciences 461 (2058), 1829–1846.
Guest, S.D., Fowler, P.W., 2005. A symmetry-extended mobility rule. Mechanism and Machine Theory 40, 1002–1014.
Graver, J.E., Servatius, B., Servatius, H., 1993. Combinatorial Rigidity. Graduate Studies in Mathematics, AMS, Providence.
Graver, J.E., 2001. Counting on Frameworks: Mathematics to Aid the Design of Rigid Structures. The Mathematical Association of America, Washington, DC.
Hendrickson, B., Jacobs, D., 1997. An algorithm for two-dimensional rigidity percolation: the pebble game. Journal of Computational Physics 137, 346–365.
James, G., Liebeck, M., 2001. Representations and Characters of Groups, second ed. Cambridge University Press.
Kangwai, R.D., Guest, S.D., 2000. Symmetry-adapted equilibrium matrices. International Journal of Solids and Structures 37, 1525–1548.
Laman, G., 1970. On graphs and rigidity of plane skeletal structures. Journal of Engineering Mathematics 4, 331–340.

Maxwell, J.C., 1864. On the calculation of the equilibrium and stiffness of frames. Philosophical Magazine 27, 294–299 (Also: Collected papers, XXVI. Cambridge University Press, 1890).

Pellegrino, S., Calladine, C.R., 1986. Matrix analysis of statically and kinematically indeterminate structures. International Journal of Solids and Structures 22, 409–428.

Schulze, B., 2008. Injective and non-injective realizations with symmetry. arXiv:0808.1761.

Schulze, B., in preparation-a. Symmetrized Laman's theorems. York University, Toronto, Canada.

Schulze, B., in preparation-b. Combinatorial and geometric rigidity with symmetry constraints. Ph.D. Thesis. York University, Toronto, Canada.

Schulze, B., Watson, A., Whiteley, W., in press. Symmetry, flatness, and necessary conditions for independence.

Tay, T.-S., Whiteley, W., 1985. Generating isostatic frameworks. Structural Topology 11, 20–69.

Whiteley, W., 1991. Vertex splitting in isostatic frameworks. Structural Topology 16, 23–30.

Whiteley, W., 2005. Counting out the flexibility of proteins. Physical Biology 2, 116–126.